# STA4246 Final Project

# Mean Field Games In
# A Stackelberg Problem with an Informed Major Player
# Bergault, Cardaliaguet, and Rainer 2023

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## 1 Abstract

This report is a review of the paper "Mean Field Games in a Stackelberg problem with an informed major player" [4]. The paper analyzes the effect that an imbalance of information has on a Stackelberg problem with a large number of small followers. In particular, the leader receives a private signal about the world and must optimize their cost, taking into account the decisions of the small players. The small players learn this signal through the major player's action. Specific assumptions on the cost functions of the players allows for the analysis of a modified information-based mean field system. The solution to this system is used to approximately solve the $N$-small player problem. This report consists of a review of the current literature, the mathematical background required to understand the analysis, and a detailed summary of the paper with some ideas for possible future extensions.

## Contents

# 2 Introduction

In order to make the best decision in a situation, information is required. In games, scenarios in which one's outcomes are affected by others' actions, acquiring information about your environment and others is particularly important. In the paper [4] Bergault, Cardaliaguet, and Rainer analyze the effects of the release of restricted information by a leader with a large number of followers in a stochastic setting. The setup of the paper is visualized in figure 1. At time $t = 0$ (the left graphic of 1) "nature" chooses a state variable denoted as $i$ according to a distribution $(p_i)$. This information is private to the major player. At time $t \in (0, T]$, for some terminal time $T$, the major player takes an action (the right graphic of 1). The small players, still affected by $i$, attempt to infer something about its realization through the action of the major player. The collective distribution of the small players denoted by $m_t$ then affects the cost of the major player. As a result of nature and the small players affecting the major player, she is incentivized to hide her knowledge by inducing some randomness into the action. A canonical example for this setting would be a scenario in which a hedge fund learns some information about the market that smaller traders would not have the capacity to observe. Assuming all the players are rational and optimizing, an equilibrium is sought for this setting.

Although a traditional equilibrium could be used to find a solution to this game, its solution would not allow for an accurate description of the situation. For instance, in the financial setting mentioned, small traders may look to the hedge fund before making financial decisions to gain information about the markets. That is, the situation has a *leader* and *followers* [3]. This gives rise to a *Stackelberg solution* to the problem. Such a solution is a collection of actions in a leader - follower game where given what the leader is playing, the followers are choosing from among their best options. The leader optimizes with respect to the worst case scenario from among the best options for followers [3, p. 133]. This can be thought of as a minimax decision rule for the leader [31]. Given that the best option set for the followers is not too complicated, this gives a robust definition of equilibrium that incorporates the sequential decision making present. The problem now becomes, how does one solve for the best decision set for the small players in this setup? Bergault, Cardaliaguet, and Rainer do this by instead looking at a corresponding mean-field game (MFG) and use its solution to approximate the optimal decisions for the small players. Before moving into their methodology, a review of the current literature and some mathematical background is required.

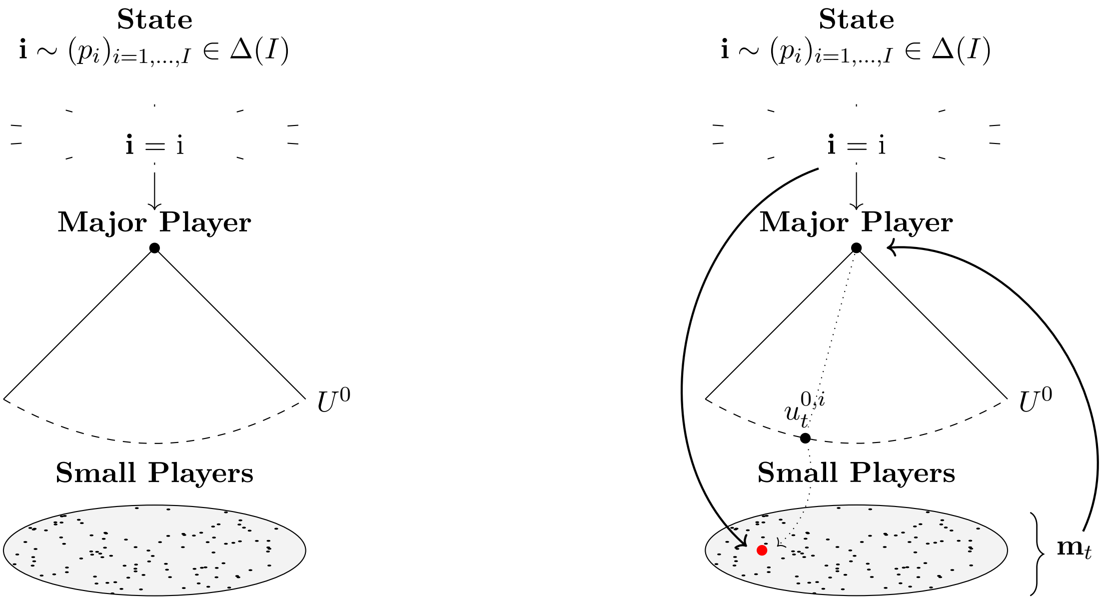


Figure 1: The Stackelberg game at time 0 and $t \in (0, T]$ respectively

# 3 Literature Review

Games with incomplete information have been studied extensively in traditional Game Theory. In static Bayesian game theory players' have types that determine outcomes [18]. Players know their own type and holds beliefs about the types of others. This uncertainty can be extended to the state of the world. Although the model Bergault et al. analyze is not explicitly framed in a static Bayesian setting, there is a valid interpretation of it as a Harsanyi type game [18] if players' action sequences were fixed at $t = 0$. In that case, "nature"'s action would only be observed by the major player and there is a single type of small player. The fixing of the action is extremely restrictive as it does not allow for the small players to update their beliefs during play. In contrast, repeated games with incomplete information (see [1]) allow for this updating. In the repeated two-player discrete games explored in [1] one player has informational advantages. Still, to get to the scenario of interest, the continuous time case must be considered. Inspired by [1], Cardaliaguet and Rainer developed a particular two player stochastic setup in 2009 [6]. In this paper the terminal costs of two players are determined by nature initially. Without knowledge of the other's realization they optimize while controlling the same state. In Section 5, Cardaliaguet and Rainer develop the theory to include the running cost of the players. This results in a setup that bears a striking similarity to the present paper. The extension to a large number of small players requires a more complicated analysis: mean field game theory.

Since their creation by Lasry-Lions [25] and Huang-Caines-Malhamé [20], mean field games have seen a lot of development and applications to various control problems [5]. However, a restriction that is present in the base assumption of an MFG (to be seen in 5) is the homogeneity of the players. One way to relax this is to include a "major" player that does not contribute to the mean field directly. However, the stochasticity of the major player necessarily adds a common noise element to the MFG system. This was originally developed in [19] and has seen use in a variety of contexts including modeling market-making problems [2], the evolution of interest rates [12], and the effectiveness of carbon emission regulation [13]. As noted in [4], there is still room for development on MFGs with imperfect information. One approach employs nonlinear filtering of agents' state processes. In [29] each agent observes some nonlinear and noisy observation of their own state. Another approach to partial information is [11] in which different sub groups of agents have different beliefs about an underlying price process. There has been some work to use the above in problems with a major player [4]. [28] explores a setting with nonlinear dynamics and costs where a major agents state is only partially observed. Under a slightly different information setup, [17] has explored the use of Kalman filters in a linear-quadratic MFG setting. However, here both the minor and major agents do not fully observe their own or each others states'. The particular MFG system used in [4] is a modification to the one introduced in [7]. To simplify the Stackelberg problem, uniqueness of the MFG solution is required. This is a complicated task considering the need for common noise as a result of the major player and the information process for the small players. However, Theorem 4.2 of [7] establishes this with some assumptions on the separability of costs shown in 7 (see B.1 for an outline). Further, the use of this equilibria as an approximation to the $N-$player setting (11.1) requires a modification to the work done in [22]. In particular the result that every weak solution of a MFG with common noise can be obtained as a limit of approximate equilibria from the corresponding $N-$player game.

In the end, the paper "Mean Field Games: A Stackelberg Problem with an Informed Major Player" extends the extant literature by considering a framework where the major player can actively manipulate her action to deceive the small players. To be able to understand their work, some mathematical background will be introduced.

# 4 Mathematical Background

Background in several areas is required to understand the tools used to find solutions of MFGs. In particular, stochastic calculus, Hamilton-Jacobi Bellman equations, and Fokker-Planck equations. Once this background has been developed, following [5] the existence and uniqueness of a simple mean field game is shown under standard assumptions.

## 4.1 Stochastic Calculus

To perform further analysis, a probability space that incorporates the dimension of time in a well-defined way is required.

**Definition 4.1.** *A* ***complete filtered probability space*** *[30] is a tuple* $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\leq T}, \mathbb{P})$ *where*

- $\Omega$ *is an arbitrary set*
- $\mathcal{F}$ *is a* $\sigma-$*algebra*
- $\mathbb{F} = (\mathcal{F}_t)_{t\leq T}$ *is a right continuous filtration i.e.*
  - $\forall t \leq T$ $\mathcal{F}_t \subset \mathcal{F}$ *are* $\sigma-$*algebras*
  - $\mathcal{F}_t \subset \mathcal{F}_s$ *if* $t \leq s$
  - $\mathcal{F}_t = \cap_{\epsilon>0}\mathcal{F}_{t+\epsilon}$
- $\mathcal{F}_0$ *contains all subsets of null sets*
- $\mathbb{P}$ *is a probability measure*

A **process** is a collection of random variables indexed by time $\{X_t\}_{0\leq t\leq T}$ defined on this space. The notion of the filtration implies other time-dependent definitions of measurabiltiy. In particular, $\{X_t\}$ is said to be $\mathbb{F}-$adapted if $\forall t \in [0, T]$ $X_t$ is $\mathcal{F}_t-$measurable [30]. On top of measurability requirements, some assumptions on the continuity of processes are useful.

**Definition 4.2.** *[21] A process $\{X_t\}$ is said to be càdlàg or right continuous left limit exists (RCLL) if $\forall \omega \in \Omega$ and $t_0 \in [0,T]$*

- $\lim_{t \searrow t_0} X_t(\omega) = X_{t_0}(\omega)$
- $\lim_{t \nearrow t_0} X_t(\omega)$ *exists*

In the main theorem of [4] the result is firstly proven for a specific filtered probability space 8.1:

**Definition 4.3.** *Firstly consider the measurable space $(C([0,\infty)), \mathcal{F})$ where $\mathcal{F}$ are the Borel sets generated by cylinder sets of the form $C = \{\omega \in C([0,\infty)) : (\omega(t_1), ...., \omega(t_n)) \in A\}$ where $A \in \mathcal{B}(\mathbb{R}^d)$. Equipping this with:*

- *The natural filtration $\mathcal{F}_t = \sigma(\omega_s : 0 \leq s \leq t)$ in addition to null sets*
- *A measure $W$, the* ***Weiner measure*** *such that*

$$B_t(\omega) = \omega(t) \text{ for } 0 \leq t < \infty \text{ is a Brownian motion}$$

*where completions to the $\sigma-$algebras yields the* ***canonical probability space*** *$(\Omega, \mathcal{F}, \mathcal{F}_t, W)$ for Brownian motion [21]*

Brownian motion is an example of an important class of processes, martingales.

**Definition 4.4.** *An adapted process $(X_t)_{t \leq T}$ is a martingale [27] if $\mathbb{E}|X_t| < \infty \; \forall t \in T$ and*

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s \text{ a.s for } 0 \leq s \leq t \leq T$$

A slightly more general version of a martingale is a semimartingale (see [27]). On this class of functions, one can integrate processes with respect to them.

**Definition 4.5.** *The stochastic integral ([27]) of a simple process with respect to a semimartingale $M$ is a random process defined as*

$$\int_0^t \alpha_s dM_s = \sum_{k=1}^n \alpha_k (M_{t_{k+1} \wedge t} - M_{t_k \wedge t})$$

*where a simple process takes the form $\alpha_t = \sum_{k=1}^n \alpha_k I_{(t_k, t_{k+1}]}(t)$*

This definition can be extended to more general processes via the density of simple processes on a Hilbert space. On semimartingales in particular, an incredibly important theorem allows one to take derivatives of functions of processes.

**Proposition 4.1.** *Given a continuous semimartingale $X$ valued in $\mathbb{R}^d$ and $f$ a function of class $C^{1,2}$ on $T \times \mathbb{R}^d$. Itö's formula ([21]) gives that $(f(t, X_t))_{t \in [0,T]}$ is a semimartingale and we have for all $t \in [0,T]$*

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(u, X_u) du + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(u, X_u) dX_u^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(u, X_u) d\langle X^i, X^j \rangle_u$$

With these tools in hand, one can look to study solutions to stochastic differential equations. Given a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ that supports a Brownian motion $W = (W_t)_{t \geq 0}$ consider the equation

$$\begin{cases} dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t & \text{for } 0 \leq t \leq T \\ X_0 = \xi \end{cases} \tag{1}$$

There are different notions of a solution to this equation. The most natural one being

**Definition 4.6.** *A **strong solution** of the above SDE starting at t is a progressively measurable process X (see [30] for the definition of progressive) lying on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that*

$$\int_t^s |b(u, X_u)|du + \int_t^s |\sigma(u, X_u)|^2 du < \infty \ a.s\ \forall t \le s \le T$$

*and the following relations:*

$$X_s = X_t + \int_t^s b(u, X_u)du + \int_t^s \sigma(u, X_u)dW_u \quad t \le s \le T$$

*holds true a.s.*

With some basic assumptions, one can guarantee the existence of such a solution.

**Theorem 4.1.** *In particular, if one assumes that a (deterministic) constant $K$ and a real-valued process $\kappa$ such that for all $t \in T$, $\omega \in \Omega$, and $x, y \in \mathbb{R}^n$*

$$|b(t, x, \omega) - b(t, y, \omega)| + |\sigma(t, x, \omega) - \sigma(t, y, \omega)| \le K|x - y|$$
$$|b(t, x, \omega)| + |\sigma(t, x, \omega)| \le \kappa_t(\omega) + K|x|$$

*where*

$$\mathbb{E}\left[\int_0^t |\kappa_u|^2 du\right] < \infty, \forall t \in T$$

*there exists for all $t \in T$, a strong solution to the SDE 1 starting at time t [30]*

It is often easier to relax the assumption that the solution lies on the same probability space. This is referred to as a 'weak' solution to 1.

**Definition 4.7.** *A weak solution (2.10 in [30]) to the SDE 1 with initial distribution $\mu \in \mathcal{P}(\mathbb{R}^m)$ is a tuple $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{X})$ where the following holds:*

- *$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ is a complete filtered probability space*
- *$\tilde{W}$ is an $(\tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ Brownian motion*
- *$\tilde{X} = (\tilde{X}_t)_{t \in [0,T]}$ is an $(\tilde{\mathbb{F}}, \tilde{\mathbb{P}})-$semimartingale*
- *$\tilde{X}_0 \sim \mu$*
- *The SDE is satisfied in this space:*

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t b(s, \tilde{X}_s)ds + \int_0^t \sigma(s, \tilde{X}_s)d\tilde{w}_s \text{ for } t \in [0, T]$$

Although the definition of weak solution appears quite broad, two weak solutions can be compared. In particular, **pathwise uniqueness** holds for 1 on $(\Omega, \mathcal{F}, \mathbb{P})$ if given two weak solutions (they may have different filtrations and brownian motion on those filtrations)

$$\mathbb{P}(X_t = \tilde{X}_t \ \ \forall 0 \le t < \infty) = 1$$

It turns out that this condition allows one to conclude strong existence of 1 using weak existence. Consider the following corollary of Yamada & Watanabe (1971) (Theorem 5.3.20 in [21]):

**Proposition 4.2.** *The existence of a weak solution to 1 and path-wise uniqueness implies strong existence of 1*

This proposition will be essential later on to prove the strong existence of a complicated MFG 8.1.

## 4.2 Hamilton - Jacobi - Bellman Equations

In the problem, small agents seek to minimize their own cost subject to their state. A simple version of this, without the interaction from others is given in the following overview of the outline in [27] and [30]:

**Definition 4.8.** *(Cost-minimization Control Problem) Choose a control $\{\alpha_s\}$ in the problem*

$$\inf_{\alpha} \mathbb{E}\left[\int_0^T L(X_s^{\alpha_s}, \alpha_s)ds + G(X_T^{\alpha_T})\right]$$

*Where $\{X_t\}_{t\in[0,T]}$ in $\mathbb{R}^d$ is a state process that follows the SDE:*

$$dX_t = b(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dB_t$$

*and $f : [0,T] \times \mathbb{R}^n \times \mathcal{A} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are the running and terminal cost functions respectively.*

The PDE approach to dynamic programming gives a way to find a solution to the above.

**Definition 4.9.** *Denote by $\mathcal{A}$ the set of control processes $\alpha$ such that*

$$\mathbb{E}\left[\int_0^T |\alpha_t(0)|^2 dt\right] < \infty$$

*and the set $\mathcal{A}(t,x) \subset \mathcal{A}$ the controls such that*

$$\mathbb{E}\left[\int_t^T |L(X_s^{t,x}, \alpha_s)|ds\right] < \infty$$

*where $X_s^{t,x}$ is the process $\{X_s\}_{s\geq t}$ conditioned such that at $t$ it is $x$.*

If one is at $x$ at time $t$, they must still choose an optimal control for the rest of the time period. The following functional referred to as the 'gain function' $J$ for $\alpha \in \mathcal{A}(t,x)$ represents this situation

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^T L(X_s^{t,x}, \alpha_s)ds + G(X_T^{t,x})\right]$$

Optimizing over the possible controls taken yields the value function:

$$v(t,x) = \inf_{\alpha\in\mathcal{A}(t,x)} J(t,x,\alpha)$$

This function may change over time, however, intuition implies that if a player is playing optimally now for the rest of the time period, they shouldn't change their mind about their control in the future. This is reflected in the following principle:

**Principle 4.1.** *For any time point $s \in [t,T]$:*

$$v(t,x) = \inf_{\alpha\in\mathcal{A}(t,x)} \mathbb{E}\left[\int_t^s L(X_u^{t,x}, \alpha)du + v(s, X_s^{t,x})\right]$$

The natural question is, what happens to this equation when $s \searrow t$? Assuming $v$ is smooth enough, Itö's (4.1) and the MVT give a workable result. Informally, consider $v$ at time $t+h$. Applying Itö's yields that

$$v(t+h, X_{t+h}^{t,x}) = v(t,x) + \int_t^{t+h} \left( \frac{\partial v}{\partial t} + \mathcal{L}^a(v) \right) + \text{ martingale}$$

Where $\mathcal{L}^a$ is the operator given by

$$\mathcal{L}^a(v) = b(x,a) \cdot D_x v + \frac{1}{2} Tr(\sigma(x,a)\sigma^T(x,a) D_x^2 v)$$

Taking the expectation of each side and $h \searrow 0$ with the MVT yields that

$$-\frac{\partial v}{\partial t}(t,x) - \inf_{a \in A}[\mathcal{L}^a v(t,x) + L(x,a)] = 0, \quad \forall (t,x) \in [0,T) \times \mathbb{R}^n \tag{2}$$

This equation can be simplified by introducing a new function.

**Proposition 4.3** (Hamilton-Jacobi Bellman Equation)**.** *Given a function $H$ defined as*

$$H(t,x,p,M) = \inf_{a \in A} \left[ b(x,a) \cdot p + \frac{1}{2} Tr(\sigma\sigma^T(x,a)M) + L(x,a) \right]$$

*called the* ***Hamiltonian****, the above becomes*

$$-\partial_t v(t,x) - H(t,x,D_x v(t,x), D_x^2 v(t,x)) = 0 \quad \forall (t,x) \in [0,T) \times \mathbb{R}^n \tag{3}$$

See [27] for a more detailed view into the verification theorem for this PDE that allows one to attain optimal controls.

## 4.3 A Simple N-Player Stochastic Game

The following four subsections are mainly taken from the excellent notes produced by Cardaliaguet [5]. Now that the theory for how a single player may optimize their costs by picking a control in accordance with their HJB equation, the question now becomes how this can applied to solve $N-$player games.

Consider a game with $N$ players. Having an individual player be represented by the index $i$ $(i = 1, ..., N)$ with a control $\alpha^i$ pushing their position process $X_t^i \in \mathbb{R}^d$ by the SDE 1 with respect to a individual Brownian motion $(B_t^i)$.

$$X_t^i = X_0^i + \int_0^t \alpha_s^i ds + \sqrt{2} B_t^i$$

It is assumed that $\forall i \; X_0^i \sim m_0$ and that each player's cost [5] is

$$\begin{aligned} &J_i(\alpha^1, ..., \alpha^N) \\ &\quad = \mathbb{E}\left[ \int_0^T \frac{1}{2}|\alpha_s^i|^2 + F\left( X_s^i, \frac{1}{N-1} \sum_{j \neq i} \delta_{X_s^j} \right) ds + G\left( X_T^i, \frac{1}{N-1} \sum_{j \neq i} \delta_{X_T^j} \right) \right] \end{aligned}$$

A tuple of games controls $(\alpha_1, ..., \alpha_N)$ is considered a solution to the above if no player can deviate to improve their outcome. Formally:

**Definition 4.10.** *The controls $(\tilde{\alpha}_1, ..., \tilde{\alpha}_N)$ are considered an open-loop Nash equilibrium (9.2 in [30]) if $\forall i \in \{1, ..., N\}$*

$$J_i^N(\tilde{\alpha}_1, ..., \tilde{\alpha}_N) \leq J_i^N(\tilde{\alpha}_1, ..., \tilde{\alpha}_{i-1}, \alpha, \tilde{\alpha}_{i+1} ..., \tilde{\alpha}_N) \quad \forall \alpha \in \mathcal{A}$$

When finding such an action tuple, difficulty arises with analyzing the effect of the distribution of the other players $\frac{1}{N-1}\sum_{j\neq i}\delta_{X_s^j}$. In particular, it is not clear how to use the above discussion on the HJB equation 7 with this empirical distribution to solve this. However, if one assumes homogeneity among the players and takes $N\to\infty$, $\frac{1}{N-1}\sum_{j\neq i}\delta_{X_s^j}$ may converge to a distribution $m\in\mathcal{P}(\mathbb{R}^d)$. It will be shown that this heuristic argument yields an approximate solution to 4.10. The next section is dedicated to analyzing the flow of the measure $m$ over time.

## 4.4 Fokker-Plank Equation

Given a vector field $b:\mathbb{R}^d\times[0,T]\to\mathbb{R}$ consider the PDE system

$$\begin{cases}\partial_t m = \Delta m + \operatorname{div}(b\cdot m) & \text{in } (0,T)\times\mathbb{R}^d\\ m(0)=m_0\end{cases}\tag{4}$$

The vector $b$ will be replaced by the derivative of the Hamiltonian $H_\xi(x,D\phi)$. It is assumed that the vector field $b:\mathbb{R}^d\times[0,T]\to\mathbb{R}^d$ is continuous, uniformly Lipschitz continuous in space, and bounded. Under these assumptions, one can consider a weak solution to the above.

**Definition 4.11.** *$m$ is a **weak solution** to the above if $m\in L^1([0,T],P_1)$ is such that, for any test function $\varphi\in C_c^\infty(\mathbb{R}^d\times[0,T))$, one has*

$$\int_{\mathbb{R}^d}\varphi(x,0)dm_0(x)+\int_0^T\int_{\mathbb{R}^d}(\partial_t\varphi(x,t)+\Delta\varphi(x,t)+\langle D\varphi(x,t),b(x,t)\rangle)dm(t)(x)=0$$

Going back to the process which the agents follow,

$$\begin{cases}dX_t^\alpha = b(X_t,t)dt+\sqrt{2}dB_t\\ X_0=Z_0\end{cases}$$

From 4.1 such assumptions on $b$ result in an unique solution to the above SDE. Further, the law of this SDE follows solves the equation in the weak sense.

**Proposition 4.4.** *If $\mathcal{L}(Z_0)=m_0$, then $m(t)=\mathcal{L}(X_t)$ is a weak solution of the system*

*Proof.* Following the definition of a weak solution, let $\varphi\in C_c^\infty(\mathbb{R}^d\times[0,T))$. Considering the process $\varphi(X_t,t)$, since $\varphi$ is $C^2$ and $C^1$ in space and time respectively, Itô's (cite Le Gall) gives that

$$\varphi(X_t,t)=\varphi(Z_0,0)+\int_0^t[\varphi_t(X_s,s)+\langle D\varphi(X_s,s),b(X_s,s)\rangle+\Delta\varphi(X_s,s)]ds+\int_0^t\langle D\varphi(X_s,s),dB_s\rangle$$

Taking expectation yields

$$\mathbb{E}[\varphi(X_t,t)]=\mathbb{E}\left[\varphi(Z_0,0)+\int_0^t[\varphi_t(X_s,s)+\langle D\varphi(X_s,s),b(X_s,s)\rangle+\Delta\varphi(X_s,s)]ds\right]$$

Since $m$ is the law of $X_t$ this just results in

$$\begin{aligned}\int_{\mathbb{R}^d}\varphi(x,t)dm(t)(x)&=\int_{\mathbb{R}^d}\varphi(x,0)dm_0(x)\\&\quad+\int_0^t\int_{\mathbb{R}^d}[\varphi_t(x,s)+\langle D\varphi(x,s),b(x,s)\rangle+\Delta\varphi(x,s)]dm(s)(x)ds\end{aligned}$$

Plugging in $t = T$ one yields that

$$0 = \int_{\mathbb{R}^d} \varphi(x,0)dm_0(x) + \int_0^T \int_{\mathbb{R}^d} [\varphi_t(x,s) + \langle D\varphi(x,s), b(x,s)\rangle + \Delta\varphi(x,s)]dm(s)(x)ds$$

Since $T$ is not in the support of $\varphi$ by assumption $\varphi(x,T) = 0$.

$\square$

This result is used to find bounds on the distance between measures A.1. This is used later in 4.2 to ensure a fixed point theorem can be applied to find a solution to the MFG system described below.

### 4.5 Mean-Field Equation

The tools have been developed to analyze the main object of consideration, the mean-field system. Consider measures $m \in \mathcal{P}_1$ with the Kantorovitch-Rubinstein distance $\mathbf{d}_1(\mu,\nu)$ 5.1. One seeks solutions $(\phi, m)$ that satisfy the following system

$$\begin{cases} \partial_t\phi = -\Delta\phi + \frac{1}{2}|D\phi|^2 - F(x,m) \quad \text{in } (0,T)\times\mathbb{R}^d \\ \partial_t m = \Delta m + \operatorname{div}(Du\cdot m) \quad \text{in } (0,T)\times\mathbb{R}^d \\ m(0) = m_0, \qquad \phi(x,T) = G(x,m(T)) \end{cases} \tag{5}$$

**Assumptions 4.1.** *To ensure existence and uniqueness of a solution, assumptions on the value functions $F$ and $G$ and the initial distribution $m_0$ are required:*

- *$F$ and $G$ are uniformly bounded by $C_0$ over $\mathbb{R}^d\times\mathcal{P}_1$ and they are Lipschitz continuous with respect to space and measure (using the Kantorovich distance)*

$$|F(x_1,m_1) - F(x_2,m_2)| \le C_0[|x_1-x_2| + \boldsymbol{d}_1(m_1,m_2)] \quad \forall (x_1,m_1),(x_2,m_2)\in\mathbb{R}^d\times\mathcal{P}_1$$
$$|G(x_1,m_1) - G(x_2,m_2)| \le C_0[|x_1-x_2| + \boldsymbol{d}_1(m_1,m_2)] \quad \forall (x_1,m_1),(x_2,m_2)\in\mathbb{R}^d\times\mathcal{P}_1$$

- *The initial measure $m_0$ has a Hölder continuous density with respect to the Lebesgue measure such that $\int_{\mathbb{R}^d}|x|^2 m_0(x)dx < \infty$*

- *Monotonicity hold for $F$ and $G$. That is,*

$$\int_{\mathbb{R}^d}(F(x,m_1)-F(x,m_2))d(m_1-m_2)(x) > 0 \quad \forall m_1,m_2\in\mathcal{P}_1, m_1\ne m_2$$
$$\int_{\mathbb{R}^d}(G(x,m_1)-G(x,m_2))d(m_1-m_2)(x) \ge 0 \quad \forall m_1,m_2\in\mathcal{P}_1$$

**Theorem 4.2** (Theorem 3.1 and 3.6 [5])**.** *Under 4.1, **there is a unique classical solution** to the MFG system 5*

Before getting to the proof of this, a few asides must be made. In particular, some results about a generalized heat equation:

**Theorem 4.3** (Theorem 5.1 of [24])**.** *Consider the heat equation*

$$\begin{cases} \partial_t w = \Delta w - \langle a(x,t), Dw\rangle - b(x,t)w + f(x,t) \quad \textit{in } \mathbb{R}^d\times[0,T] \\ w(x,0) = w_0(x) \quad \textit{in } \mathbb{R}^d \end{cases} \tag{6}$$

*with $a : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, $b, f : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ and $w_0 : \mathbb{R}^d \to \mathbb{R}$ belonging to $\mathcal{C}^\alpha$ for $\alpha \in (0,1)$, then the above heat equation has a unique weak solution in $\mathcal{C}^{2+\alpha}$. Further, if $a = b = 0$ and $f$ is continuous and bounded, any classical, bounded solution $w$ of the above satisfies for any compact $K \subset \mathbb{R}^d \times (0,T)$,*

$$\sup_{(x,t),(y,s)\in K} \frac{|D_x w(x,t)| + D_x w(y,s)}{|x-y|^\beta + |t-s|^{\beta/2}} \leq C(K, \|w\|_\infty)\|f\|_\infty$$

*where $\beta$ depends on the dimension $d$ and $C(K, \|w\|_\infty)$ on the compact set $K$, $d$, and $\|w\|_\infty$*

The essence of the proof 4.2 is to create a mapping from measures to measures that involves the heat equations present in the system. 4.3 will be used to ensure the map is well defined. This map will be on the following set of measures:

**Definition 4.12.** *Given some constant $C_1$ let $\mathcal{K}$ be the set of maps $\mathcal{C}^0([0,T], \mathcal{P}_1)$ such that*

$$\sup_{s\neq t} \frac{\mathbf{d}_1(\mu(s), \mu(t))}{|t-s|^{\frac{1}{2}}} \leq C_1$$

*and*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} |x|^2 dm(t)(x) \leq C_1$$

*From A.1 and lemma 5.7 of [5] $\mathcal{K}$ is a convex, closed, and compact subset of $\mathcal{C}^0([0,T], \mathcal{P}_1)$.*

Using the above results and definitions the proof of existence and uniqueness to the MFG 5 can be started.

*Proof of Theorem 4.2.* We begin by defining a mapping from $\mathcal{K}$, $\Psi$. Let $\mu \in \mathcal{K}$. Let $\phi$ be the unique solution to the equation

$$\begin{cases} \partial_t \phi = -\Delta\phi + \dfrac{1}{2}|D\phi|^2 - F(x, \mu(t)) & \text{in } (0,T)\times\mathbb{R}^d \\ \phi(x,T) = G(x, \mu(T)) & \text{in } \mathbb{R}^d \end{cases} \tag{7}$$

With this define $m = \Psi(\mu)$ as the unique solution to

$$\begin{cases} \partial_t m = \Delta m + \operatorname{div}(D\phi \cdot m) & \text{in } (0,T)\times\mathbb{R}^d \\ m(0) = m_0 & \text{in } \mathbb{R}^d \end{cases} \tag{8}$$

We now aim to show that $\Psi$ is well-defined and continuous. We do this with two applications of 4.3 on the above systems to ensure uniqueness of the output of $\Psi$.
Firstly, consider the transformation $w = e^{\frac{\phi}{2}}$ of $\phi$. Then

$$\begin{aligned}
-\partial_t w - \Delta w &= -\partial_t\left(e^{\frac{\phi}{2}}\right) - \sum_{i=1}^d \partial^2_{x_i}\left(e^{\frac{\phi}{2}}\right) \\
&= -\frac{1}{2}e^{\frac{\phi}{2}}\partial_t\phi - \sum_{i=1}^d \partial_{x_i}\left(\tfrac{1}{2}e^{\frac{\phi}{2}}\partial_{x_i}\phi\right) \\
&= -\tfrac{1}{2}w\,\partial_t\phi - \sum_{i=1}^d\left(\tfrac{1}{4}w(\partial_{x_i}\phi)^2 + \tfrac{1}{2}w\,\phi_{x_ix_i}\right) \\
&= -\tfrac{1}{2}w\,\partial_t\phi - w\left(\tfrac{1}{4}\sum_{i=1}^d \phi^2_{x_i} + \tfrac{1}{2}\sum_{i=1}^d \phi_{x_ix_i}\right) \\
&= w\left(-\phi_t - \tfrac{1}{4}|\nabla\phi|^2 - \tfrac{1}{2}\Delta\phi\right) \\
&= \frac{-w}{2}\left(2\phi_t + \Delta\phi + \tfrac{1}{2}|\nabla\phi|^2\right)
\end{aligned}$$

Through a rescaling of time in the input of $\phi$ and the fact that it satisfies 7 yields that

$$\begin{cases} -\partial_t w = \Delta w - wF(x, \mu(t)) & \text{in } \mathbb{R}^d \times [0, T] \\ w(x, T) = e^{G(x, \mu(T))/2} & \text{in } \mathbb{R}^d \end{cases} \tag{9}$$

Now, since $\mu \in \mathcal{K}$ and $F$ and $G$ satisfy the Lipschitz conditions outlined in 4.1. We get that $F(x, m(t))$ and $e^{G(x, \mu(T)/2)}$ are inside $\mathcal{C}^\alpha$ where $\alpha = \frac{1}{2}$. Thus, Theorem 4.3 can be applied to find that there is a unique $w$ which satisfies this equation and is inside $\mathcal{C}^{2+\alpha}$. Thus a unique $\phi$ exists that satisfies the above and is inside $\mathcal{C}^{2+\alpha}$.
We wish to apply 4.3 to the above Fokker-Planck equation. Firstly, it must be noted that the Fokker-Planck equation can be written in the form:

$$\partial_t m - \Delta m - \langle Dm, D\phi(t, x) \rangle - m\Delta\phi(x, t) = 0$$

Since $\phi \in \mathcal{C}^{2+\alpha}$ the maps (by definition of Holder continuity) $a(x, t) = D\phi(x, t)$ and $b(x, t) = \Delta\phi(x, t)$ belong to $\mathcal{C}^\alpha$. Thus, 4.3 can be applied again to find that this is uniquely solvable and $m \in \mathcal{C}^{2+\alpha}$. So, $\Phi(\mu)$ is well defined. We wish to show that $\mu \in \mathcal{K}$ to begin setup a use of A.1. In the above analysis for $\phi$ an application of the comparison principle yields that $D\phi$ is bounded by the Lipschitz constant $C_0$. Combining this with lemma A.1 yields that $m \in \mathcal{K}$ by its definition. Thus, $\Psi : \mathcal{K} \to \mathcal{K}$. We now wish to show $\Psi$ is continuous to apply A.1.

Let $\{\mu_n\} \subset \mathcal{K}$ converge to some $\mu$ (note by $\mathcal{K}$'s closedness $\mu \in \mathcal{K}$) that is, $\mathbf{d}_1(\mu_n, \mu) \to 0$. From each of the steps above, we let $(\phi_n, m_n)$ and $(\phi, m)$ be the corresponding solutions. Note that the mappings $(x, t) \to F(x, \mu_n(t))$ and $x \to G(x, \mu_n(T))$ locally uniformly converge to $(x, t) \to F(x, \mu(t))$ and $x \to G(x, \mu(T))$. Then, one gets the local uniform convergence of $(u_n)$ to $u$ by standard arguments [5]. Since the $(D_x\phi_n)$ are uniformly bounded, the $(\phi_n)$ solve an equation of the form

$$\partial_t u_n - \Delta u_n = f_n$$

where $f_n = \frac{1}{2}|D_x\phi_n|^2 - F(x, m_n)$ is uniformly bounded in $x$ and $n$. The interior regularity from 4.3 implies that $(D_x u_n)$ is locally uniformly Hölder continuous and therefore locally uniformly converges to $D_x u$. This implies that any converging subsequence of the relatively compact $m_n$ is a weak solution of 8, but $m$ is the unique weak solution of 8 and so $(m_n)$ converges to $m$.
Thus, the map $\Psi$ is continuous and so since $\mathcal{K}$ is compact and convex A.1 can be applied to find that a fixed point exists and is a solution of the system. The proof on uniqueness relies on truncating and regularizing the difference between two solutions of the system and applying the weak formulation of the HJB and Fokker Planck equation. This combined with the monotonicity assumption implies equality of the solutions. Please see [5] Theorem 3.6 for more detail.

□

Looking back to the original problem of finding equilibria 4.10 in the $N-$player game, it turns out that this unique solution $(\phi, m)$ is useful in approximating Nash equilibria for large enough $N$.

## 4.6 Application to N-Player game

Using the same notation as in the $N-$player game section, the optimal action $\tilde{\alpha}$ it provides is approximately a 4.10. This is given in the following theorem:

**Theorem 4.4** ($\delta-$Nash Equilibria, 3.8 in [5])**.** *$\forall \delta > 0$, there is some $N_0$ such that, if $N \geq N_0$ then in the $N-$player game described above $(\tilde{\alpha}^1, ..., \tilde{\alpha}^N)$ is a $\delta-$Nash equilibrium in the sense that*

$$J_i^N(\tilde{\alpha}_1, ..., \tilde{\alpha}_N) \leq J_i^N(\tilde{\alpha}_1, ..., \tilde{\alpha}_{i-1}, \alpha, \tilde{\alpha}_{i+1} ..., \tilde{\alpha}_N) + \delta$$

*for any control* $\alpha$ *and* $i \in \{1, ..., N\}$

This is the main use of mean field Systems. From a relatively simple PDE system, approximate solutions to large player games can be found. Essentially the same process will be used to find an approximate solution to the $N-$player Stackelberg game.

# 5 Some Definitions

Before getting into the exact methodology of the paper, some definitions and assumptions that will be used are presented.

**Definition 5.1.** *The set* $\mathcal{P}_2(\mathbb{R}^d) \subset \mathcal{P}(\mathbb{R}^d)$ *is the set of measures* $\mu$ *such that*

$$\int_{\mathbb{R}^d} |x|^2 d\mu(x) < \infty$$

*On this space, the measure* $\mathbf{d}_2 : \mathcal{P}_2(\mathbb{R}^d)^2 \to \mathbb{R}^+$ *is defined as*

$$\mathbf{d}_2(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^d} |x - y|^2 d\gamma(x, y)$$

*where* $\Gamma(\mu, \nu) \subset \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ *is the set of all couplings of* $\mu$ *and* $\nu$*. Other metrics are defined in the same way, in particular* $\mathbf{d}_1$*.*

However, unless otherwise specified, the work in this paper is done within $\mathcal{P}(\mathbb{R}^d)$

**Definition 5.2.** *Fixing an* $I \in \mathbb{N}^+$ *consider the simplex* $\Delta(I)$ *defined as*

$$\Delta(I) = \left\{ p \in \mathbb{R}^I_+ : \sum_i p_i = 1 \right\}$$

**Definition 5.3.** *A Polish space* $\mathbf{P}$ *[15] is a topological space* $(\mathbb{X}, \mathcal{T})$ *that is separable and complete*

**Definition 5.4.** $\mathbf{D}$ *is defined to be the set of cadlag functions from* $\mathbb{R} \to \Delta(I)$*, endowed with the Meyer-Zheng topology [26]*

The following space will be used to define a relaxation of the major player's problem.

**Definition 5.5.** *On* $\mathbf{D}$ *endow a measurable space with the Borel* $\sigma$*-algebras induced by the Meyer-Zheng topology on the coordinate mappings. Now, given a* $p_0 \in \Delta(I)$*, denote by* $\mathbf{M}(p_0)$ *the set of probabilities over this measurable space such that* $(\mathbf{p}(t) : t \in \mathbb{R})$ *is a martingale with* $p_t = p_0$

The major player will have a choice over this space. The small will have an initial guess $p_0$ for the probability distribution of $\mathbf{i}$. The martingale property above ensures that this is consistent with the rationality of the small players. See 10.2 for more detail on how this space is used.

# 6 The Game

Prior to an explanation of the Stackelberg problem, the players' cost function are defined for each realization $i \in \{1, ..., I\}$ of nature.

$$L_i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

is the running cost for a small player to play a control and the interaction costs are denoted as

$$F_i, G_i : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$$

The **major player's** cost is not assumed separable between the action and distribution element. Further, it is denoted by a 0. That is,

$$L_i^0 : [0, T] \times U^0 \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$$

is the running cost of the major player. Later in the analysis the players will be concerned with the expected value of these costs given a distribution over $I$. That is, given a $p \in \Delta(I)$ and $x, u, \xi \in \mathbb{R}^d$ and distribution $m \in \mathcal{P}(\mathbb{R}^d)$ introduce another variable into the costs such that for the small players

$$L(x, u, p) = \sum_{i=1}^{I} p_i L_i(x, u), \quad F(x, m, p) = \sum_{i=1}^{I} p_i F_i(x, m), \quad G(x, m, p) = \sum_{i=1}^{I} p_i G_i(x, m)$$

Similarly for the major player:

$$L^0(s, u^0, p, m) = \sum_{i=1}^{I} p_i L_i^0(s, u^0, p, m) \text{ with the optimized } \bar{L}^0(s, p, m) = \inf_{u^0 \in U^0} L^0(s, u^0, p, m)$$

Further, the Hamiltonian for the small players, as a result of the separation present in their costs is given as

$$H(x, \xi, p) = \sup_{u \in \mathbb{R}^d} -\xi \cdot u - L(x, u, p)$$

where $\xi \in \mathbb{R}^d$ is an adjoint variable.

The game can now be mathematically formalized. The randomness present is encapsulated by nature sampling $\mathbf{i} \sim (\{1, ..., I\}, p^0)$ where $p^0 = (p_i^0)_{i=1,...,I}$.

- The goal of the major player is to then minimize over her random control denoted by $(\mathbf{u}^0 = (\mathbf{u}_i^0)_{i=1,...,I})$ a cost of the form

$$\mathbb{E}\left[\int_0^T L_{\mathbf{i}}^0(t, \mathbf{u}_{i,t}^0, \mathbf{m}_t) \mid \mathbf{i} = i\right]$$

  where $\mathbf{m}_t$ is the (random) distribution of the small players

- Observing the realization of $\mathbf{u}_{\mathbf{i}}^0$ of the informed player and their own state, the minor players aim to minimize their cost using their control $(\alpha_t)$

$$\mathbb{E}\left[\int_0^T L_{\mathbf{i}}(X_t, \alpha_t) + F_{\mathbf{i}}(X_t, \mathbf{m}_t) dt + G_{\mathbf{i}}(X_T, \mathbf{m}_T)\right]$$

  where $(\alpha_t)$ is the control of a typical small player and their state is given by the process

$$X_t = Z + \int_0^t \alpha_s ds + \sqrt{2} B_t$$

  for $t \in [0, T]$ where $Z \sim m_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and $B$ is a standard $d-$dimensional Brownian motion with $Z, B$ and $(\mathbf{i}, \mathbf{u}^0)$ independent.

In order to begin to solve the above using the tools developed thus far, many assumptions will be required to use theorems developed in past work. The next section will outline some of these assumptions.

# 7 Cost Function Assumptions

## 7.1 Major Player Assumptions

The set in which controls for the major player lies $(U^0, d^0)$ is a compact convex subset of a finite dimensional space. Further, for $i = 1, ..., I$,

$$L_i^0 : [0,T] \times U^0 \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R} \text{ is continuous and bounded}$$

## 7.2 Small Player Assumptions

For each $i$, the cost functions $F_i, G_i : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ are Lipschitz continuous and bounded. Further,

$$\sup_{m \in \mathcal{P}(\mathbb{R}^d)} \|F_i(\cdot, m)\|_{C^{2+\alpha}} + \|G_i(\cdot, m)\|_{C^{2+\alpha}} \leq C \text{ for some } C, \alpha > 0$$

Further, another set of assumptions which are intrinsic to proving uniqueness of solutions later on is that $F_i, G_i$ are strongly monotone meaning $\exists \alpha > 0$ such that

$$\int_{\mathbb{R}^d} (Q(x, m_1) - Q(x, m_2))(m_1 - m_2)(dx) \geq \alpha \int_{\mathbb{R}^d} (K(x, m_1) - K(x, m_2))^2 dx$$

for $Q = F_i$ or $G_i$ for each $i$. And $F_i$ is strictly monotone meaning

$$\int_{\mathbb{R}^d} (F_i(x, m_1) - F_i(x, m_2))(m_1 - m_2)(dx) \leq 0 \Rightarrow m_1 = m_2$$

for each $i = 1, ..., I$.

To exploit much of the work done in previous papers, in particular [7], regularity is required of the small players' Hamiltonians. That is,

- $\exists C > 0$ such that, $\forall x, \xi \in \mathbb{R}^d$,

$$C^{-1}|\xi|^2 - C \leq H_i(x, \xi) \leq C(|x|^2 + 1)$$

- $\forall R > 0$, $\mathbb{R}^d \times B_R \ni (x, \xi) \to H_i(x, \xi)$ is uniformly bounded and Lipschitz, $\xi \to H_i(x, \xi)$ is uniformly convex
- $\forall R > 0$ $\|H(\cdot, \cdot)\|_{C^{2+\alpha}(\mathbb{R}^d \times B_R)} \leq C_R$
- For some $\lambda_0, C_0 > 0$ and all $t \in [0, T]$, $\xi, \xi' \in \mathbb{R}^d$ and $|z| = 1$ we have

$$|D_x H(x, \xi)| \leq C_0 + \lambda_0(\xi \cdot D_\xi H(x, \xi) - H(x, \xi))$$
$$\lambda_0(D_\xi H(x, \xi) \cdot \xi - H(x, \xi)) + D^2_{\xi\xi} H(x, \xi)\xi' \cdot \xi' + 2D^2_{\xi x} H(x, \xi) z \cdot \xi' + D^2_{xx} H(x, \xi) z \cdot z \geq -C_0$$

**Example 7.1.** *The function $H(x, \xi) = a(x)|\xi|^2$ satisfies the above regularity conditions.*

# 8 The Information Based Mean Field System

With the above assumptions, the main driver of results in the paper can be introduced. Firstly, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{P})$ be filtered probability space. To incorporate the information gained by the small players, consider a process

$$p = (p_t) \subset \Delta(I) \text{ adapted to } (\mathcal{F}_t)$$

With this, consider the modified MFG system:

$$\begin{cases} d\phi_t(x) = \{-\Delta\phi_t(x) + H(x, D\phi_t(x), p_t) - F(x, m_t, p_t)\}dt + dM_t(x) & \text{in } (0,T)\times\mathbb{R}^d \\ dm_t(x) = \{\Delta m_t(x) + div(H_\xi(x, D\phi_t(x), p_t)m_t(x))\}dt & \text{in } (0,T)\times\mathbb{R}^d \\ m_0(x) = \bar{m}_0 s(x), \quad \phi_T(x) = G(x, m_T, p_T) & \text{in } \mathbb{R}^d \end{cases} \tag{10}$$

There are two main differences between this and 5. Firstly, the generalization to a more complicated Hamiltonian for the HJB is made. In addition, the process $p_t$ is featured as an input to the Hamiltonian in the HJB and Fokker-Planck equation. Secondly, there is a separate stochastic element $M_t$ affecting the system. This results in the system becoming a stochastic partial differential equation. It can be interpreted as the common noise felt by the small agents from the random actions of the major player. This system is a modified version of systems seen in past work. The existence of a solution to the common noise MFG was shown in [10] on a Torus and in a separated Hamiltonian case. This was further expanded on in [7] and [9]. The proof of existence is heavily influenced by the work done in [7]. The separability of the Hamiltonian above is key for the main proof 8.1. The solution of the system is the following:

**Definition 8.1.** *A triple $(\phi, m, M)$ is a solution to 14 on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ if*

1. *$\phi : [0,T]\times\Omega \to C(\mathbb{R}^d)$ is a cadlag process adapted to $(\mathcal{F}_t)$ with $\phi_T(\cdot) = G(\cdot, m_T)$*
2. *$M : [0,T]\times\Omega \to \mathcal{M}_{loc}(\mathbb{R}^d)\cap W^{-1,\infty}(\mathbb{R}^d)$ is a cadlag martingale w.r.t $(\mathcal{F}_t)$ starting at 0*
3. *$m : [0,T]\times\Omega \to \mathcal{P}_2(\mathbb{R}^d)$ is a continuous process adapted to the filtration $(\mathcal{F}_t)$, with $m_0 = \bar{m}_0$ such that $m_t$ has a bounded density on $\mathbb{R}^d$ $\mathbb{P}-a.s$ and for any $t\in[0,T]$*
4. *$\exists C > 0$ such that, with probability 1, and for a.e. $t \in [0,T]$*

$$\|D\phi_t\|_\infty + ess\sup m_+(D^2\phi_t) + \|M_t\|_{\mathcal{M}_{loc}\cap W^{-1,\infty}} + \|m_t\|_\infty \le C$$

5. *$(\phi, M)$ satisfies, in the distributional sense on $\mathbb{R}^d$, for all $t\in[0,T]$ and $\mathbb{P}-a.s.$ the equality:*

$$\phi_t(x) = G(x, m_T, p_T) + \int_t^T (\Delta\phi_s(x) - H(x, D\phi_s(x), p_s) + F(x, m_s, p_s))ds + M_t(x) - M_T(x)$$

6. *$\mathbb{P}$-a.s. and in the sense of distributions, $m$ sovles the equation*

$$dm_t(x) = \{\Delta m_t(x) + div(H_\xi(x, D\phi_t(x), p_t)m_t(x))\}dt \quad \text{in } (0,T)\times\mathbb{R}^d$$

This is seen to be exactly a regularized combined version of definitions 4.1 and 3.1 in [7].

**Theorem 8.1.** *Under the assumptions outlined in 7,* ***there exists a unique solution*** $(\phi, m, M)$. *Further, $D^2\phi_t$ and $M_t$ are absolutely continuous with Radon-Nikodym derivative bounded in $L^\infty$ $\mathbb{P}-a.s.$ and for every $t\in[0,T]$,*

$$\|D^2\phi_t\|_\infty + \|M_t\|_\infty \le C$$

*In addition, if $p$,$\tilde{p}$ have the same law on $\mathbf{D}$ and $(\phi, m, M)$ and $(\tilde{\phi}, \tilde{m}, \tilde{M})$ are the associated solutions to the system, then $(m,p)$ and $(\tilde{m}, \tilde{p})$ have the same law on $C^0([0,T], \mathcal{P}_2(\mathbb{R}^2))\times\mathbf{D}$*

*Proof Expanded Outline.* A brief sketch of this proof is given in the text, however, there are several substantial steps that are missed that should be mentioned. The proof of this theorem hinges on the substantial work done in [7] (see 15 in the appendix). In particular, Theorem 4.1 and 4.2. There are four main steps to the proof:

1. Firstly, the authors prove weak existence on the canonical space for Brownian motion 4.3. This is done through a discretization of time to dampen the common noise's effect.

2. Each discretization is treated as its own MFG and solved in a fixed point method similar to that seen in 5 (see Lemma 4.2 and 4.3 in [7])

3. Lemma 4.5 takes the discretization to the limit and the proof of Theorem 4.1 establishs that the limit is a solution to the MFG

4. Lastly, as a result of the strict monotonicity assumption in 7 and the **separability** of the cost functions Theorem 4.2 shows that pathwise uniqueness holds for the system, which when combined with the above existence of a weak solution and the Yamada Watanabe theorem 4.2 implies the existence of the strong solution outlined in 8.1

As given in the text, the necessary bounds and regularity are proved using a representation with a heat kernel. □

The following proposition gives that the abstract HJB equation in 14 corresponds to a minimization problem. In particular, the exact minimization problem the paper wishes to solve given an arbitrary process $(p_t)$.

**Proposition 8.1.** *Let $(m_t)$ be a continuous random process taking values in $\mathcal{P}_2(\mathbb{R}^d)$, adapted to the filtration $(\mathcal{F}_t)$. Then the HJ equation*

$$\begin{cases} d\phi_t(x) = \{-\Delta\phi_t(x) + H(x, D\phi_t(x), p_t) - F(x, m_t, p_t)\}dt + dM_t(x) & \text{in } (0,T)\times\mathbb{R}^d \\ \phi_T(x) = G(x, m_T, p_T) \quad \text{in } \mathbb{R}^d \end{cases} \tag{11}$$

*has a unique solution in the sense described above. Let $(\Omega^1, \mathbb{F}^1, \mathbb{P}^1, (\mathcal{F}_t^1))$ be another filtered probability space supporting a Brownian motion $B$ and a random variable $Z$ of law $\bar{m}_0$ on $\mathbb{R}^d$ and $\alpha^*$ and $X^*$ being given by*

$$X_t^* = Z - \int_0^t H_\xi(X_s^*, D\phi_s(X_s^*), p_s)ds + \sqrt{2}B_t, \quad \alpha_t^* = -H_\xi(X_t^*, D\phi_t(X_t^*), m_t, p_t), \quad t \in [0,T]$$

*Then, for any control $\alpha \in L^2((0,T)) \times \Omega \times \Omega_1)$ adapted to the filtration generated by $(p, m, B)$,*

$$\begin{aligned} \mathbb{E}^{\mathbb{P}\otimes\mathbb{P}^1}[\phi_0(Z)] &= \mathbb{E}^{\mathbb{P}\otimes\mathbb{P}^1}\left[\int_0^T (L(X_s^*, \alpha_s^*, p_s) + F(X_s^*, m_s, p_s))ds + G(X_T^*, m_T, p_T)\right] \\ &\leq \mathbb{E}^{\mathbb{P}\otimes\mathbb{P}^1}\left[\int_0^T (L(X_s, \alpha_s, p_s) + F(X_s, m_s, p_s))ds + G(X_T^*, m_T, p_T)\right] \\ &\quad - C^{-1}\mathbb{E}^{\mathbb{P}\otimes\mathbb{P}^1}\left[\int_0^T |\alpha_s + H_\xi(X_s, D\phi_s(X_s), p_s)|^2 ds\right] \end{aligned}$$

*where $X_t = Z + \int_0^t \alpha_s ds + \sqrt{2}B_t$ for $t \in [0,T]$*

The above will be used to show the uniqueness of the MFG equilibrium (small player problem) later on.

# 9 The Stackelberg Problem

The majority of the heavy lifting has been completed in 8.1 and 8.1 to solve the Stackelberg MFG game outlined in the introduction ([3], 133). This section of the paper outlines its use.

In reality, the major player must randomize her controls to some extent to hide their knowledge of the realization of nature. If this wasn't the case, the small players would be able to immediately figure out what value nature takes and optimize accordingly, possibly negatively affecting the major player. In a financial setting, a hedge fund may not want to immediately liquidate their holdings in one stock after realizing it is overvalued. A random liquidation of various shares over time with an overall goal of dropping that one stock would likely be optimal. This is indeed the setup of the Stackelberg problem. The major player will choose among a collection of random controls in order to possibly mislead the small players.

**Definition 9.1.** *Starting at a time point $t_0 > 0$ the set $\mathcal{U}^0$ are the measurable maps $u^0 : [t_0, T] \to U^0$, endowed with the $L^1-$distance*

$$d(u^0, v^0) = \int_{t_0}^{T} d^0(u_s^0, v_s^0)ds \quad \forall u^0, v^0 \in \mathcal{U}^0([t_0, T])$$

*where $\mathcal{U}^0 := \mathcal{U}^0([0,T])$ endowing it with the Borel $\sigma-$algebra generated by the above metric, and the natural filtration $(\mathcal{F}'_{\sqcup})_{t\in[0,T]} = (\sigma(s \to u_s^0 : s \leq t)_t)_{t\in[0,T]}$ $\Delta(\mathcal{U}^0)$ is the set of probabilities on this space. With this set of probabilities defined, the major player has a choice over*

$$\mathbf{u}^0 = (\mathbf{u}_i^0)_{i=1,\dots,I} \in (\Delta(\mathcal{U}^0))^I$$

The major players choice is random, but depends on the realization of nature. Thus, the action $\mathbf{u}^0$ generates the probability $P^{\mathbf{u}^0}$ observed by the small players on $(\Omega^0 \times \mathcal{U}^0, \mathcal{B}(\Omega^0 \times \mathcal{U}^0))$ defined as

$$P^{\mathbf{u}^0}(\{i\} \times A^0) = p_i^0 \mathbf{u}_i^0(A) \quad \forall i \in \{1, ..., I\}, A \in \mathcal{B}(\mathcal{U})$$

The small players will observe the realization of the major player $s \to u_s^0$. Assuming that their decisions are on a probability space $(\Omega^1, \mathbb{F}^1, \mathbb{P}^1, (\mathcal{F}_t^1))$ supporting $B$ and a random initialization $Z$ on $\mathbb{R}^d$ of law $\bar{m}_0$, the joint filtered probability space which all players will be defined on is given by $(\Omega^0 \times \mathcal{U} \times \Omega^1, \mathcal{B}(\Omega^0 \times \mathcal{U}) \otimes \mathbb{F}^1, (\mathcal{F}_t^{\mathbf{u}^0,1}), P^{\mathbf{u}^0} \otimes \mathbb{P}^1)$. Is is important to note that the joint filtration $\mathbb{F}^1, (\mathcal{F}_t^{\mathbf{u}^0,1})$ is generated by the realizations of $t \to (u_t^0, B_t)$. And so independence of the processes would lead to a slight simplification of this space.

With this defined, in accordance with the discussion of Stackelberg equilibria in the introduction, the major player's problem is

**Definition 9.2.** *The major player problem is*

$$\inf_{\mathbf{u}^0 \in (\Delta(\mathcal{U}^0))^I} J^0(\mathbf{u}^0) = \inf_{\mathbf{u}^0 \in (\Delta(\mathcal{U}^0))^I} \left( \sup_{(\alpha^{\mathbf{u}^0}, m^{\mathbf{u}^0})} \mathbb{E}^{P^{\mathbf{u}^0} \otimes \mathbb{P}^1} \left[ \int_0^T \sum_{i=1}^{I} p_i^0 L_i^0(t, u_s^0, m_s^{\mathbf{u}^0}) ds \right] \right) \tag{12}$$

*where $(\alpha^{\mathbf{u}^0}, m^{\mathbf{u}^0})$ are the MFG equilibria associated to $\mathbf{u}^0$*

That is, the major player is facing the worst case scenario of possible equilibria of the small players. Where an equilibria of the small players is defined as

**Definition 9.3** (MFG Equilibrium)**.** *Given a control $\mathbf{u}^0 \in (\Delta(\mathcal{U}^0))^I$, and MFG equilibrium associated to $\mathbf{u}^0$ is a pair $(\alpha^{\mathbf{u}^0}, m^{\mathbf{u}^0})$ of processes on $(\Omega^0 \times \mathcal{U} \times \Omega^1, \mathcal{B}(\Omega^0 \times \mathcal{U}) \otimes \mathbb{F}^1, (\mathcal{F}_t^{\mathbf{u}^0,1}), P^{\mathbf{u}^0} \otimes \mathbb{P}^1)$ where $\alpha^{\mathbf{u}^0}$ takes values in $\mathbb{R}^d$, and $m^{\mathbf{u}^0}$ in $\mathcal{P}_2(\mathbb{R}^d)$, and*

1. $\alpha^{\mathbf{u}^0}$ *is optimal in the control problem*

$$\inf_{\alpha} \mathbb{E}^{P^{\mathbf{u}^0} \otimes \mathbb{P}^1} \left[ \int_0^T (L_{\mathbf{i}}(X_t^\alpha, \alpha_t) + F_{\mathbf{i}}(X_t^\alpha, m_t^{\mathbf{u}^0}))dt + G_{\mathbf{i}}(X_T^\alpha, m_T^{\mathbf{u}^0}) \right]$$

*where the infimum is taken over all* $\mathbb{R}^d-$*valued* $(\mathcal{F}_t^{\mathbf{u}^0,1})-$*adapted controls* $\alpha$ *and*

$$X_t^\alpha = Z + \int_0^t \alpha_s ds + \sqrt{2}B_t \quad t \in [0,T]$$

2. *for any* $t \in [0,T]$, $m_t^{\mathbf{u}^0}$ *is the conditional law of* $X_t^{\alpha^{\mathbf{u}^{P_0}}}$ *given* $\sigma(\{s \to \mathbf{u}_s^0 : s \leq t\})$

It is important to note that unlike other work in this area[14], the major agent must face the worst case scenario based on the mean field equilibria induce by their action [4]. It turns out that by a clever choice of information process $(p_t)$ in 14 the next section and an application 8.1 and 8.1 from the last section, one can ensure a unique MFG equilibrium exists.
Firstly, it will be shown that the best (and only case if rational and optimizing) for the small players is one in which the outcome is given by the solution to 14 with a specific information process. In the Stackelberg setting, the small players know the optimal control $\mathbf{u}^0$ and observe the realization $\mathbf{u}_{\mathbf{i}}^0$. Thus, the small players have access to their 'best guess martingale':

$$\begin{aligned}
\mathbf{p}_t^{\mathbf{u}^0} &= \mathbb{E}^{\mathbf{u}^0} \left[ e_{\mathbf{i}} | \mathcal{F}_t^{\mathbf{u}^0} \right] \\
&= \left( \mathbb{P}^{\mathbf{u}^0}(\mathbf{i} = 1 | \mathcal{F}_t^{\mathbf{u}^0}), ..., \mathbb{P}^{\mathbf{u}^0}(\mathbf{i} = I | \mathcal{F}_t^{\mathbf{u}^0}) \right)
\end{aligned}$$

From Probability theory, we know that the $(\mathbf{p}_t^{\mathbf{u}^0})$ is a $\mathcal{F}_t^{\mathbf{u}^0}-$martingale and instead consider its cadlag 4.2 version ([21] Theorem 1.3.13). Heuristically this process is the "best" guess for the small players. An important result is that independence between the action of the major player and the process of the small player allows for this martingale to connect 14 and 9.3. Although simple, the following Lemma allows for the use of properties of the system 14.

**Lemma 9.1.** *Fix a control for the major player* $\mathbf{u}^0$ *and let* $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1, (\mathcal{F}_t^1))$ *be a filtered space supporting a Brownian motion* $B$ *and a random variable* $Z$ *on* $\mathbb{R}^d$ *of law* $\bar{m}_0$*. Let* $(m_t)$ *be a random distribution of the players i.e. a* $\mathcal{P}_2(\mathbb{R}^d)-$*valued and* $(\mathcal{F}_t^{\mathbf{u}^0})-$*adapted process. Then for any* $(\mathcal{F}_t^{\mathbf{u}^0,1})-$*adapted control* $\alpha$ *and if*

$$X_t^\alpha = Z + \int_0^t \alpha_s ds + \sqrt{2}B_t, \quad t \in [0,T]$$

*then*

$$\begin{aligned}
&\mathbb{E}^{P^{\mathbf{u}^0} \otimes \mathbb{P}^1} \left[ \int_0^T (L_{\mathbf{i}}(X_t^\alpha, \alpha_t) + F_{\mathbf{i}}(X_t^\alpha, m_t))dt + G_{\mathbf{i}}(X_T^\alpha, m_T) \right] \\
&\quad = \mathbb{E}^{P^{\mathbf{u}^0} \otimes \mathbb{P}^1} \left[ \int_0^T (L(X_t^\alpha, \alpha_t, \mathbf{p}_t^{\mathbf{u}^0}) + F(X_t^\alpha, m_t, \mathbf{p}_t^{\mathbf{u}^0}))dt + G(X_T^\alpha, m_T, \mathbf{p}_T^{\mathbf{u}^0}) \right]
\end{aligned}$$

*Proof.* This follows immediately from the independence of $(\mathcal{F}_t^{\mathbf{u}^0})$ and $(\mathcal{F}_t^1)$ and the definitions of the cost functions given in 6 □

With this connection, the results from 8.1 are applied to show that there is in fact a unique MFG equilibrium 9.3:

**Corollary 9.1.** *Under the same assumptions, given a control* $\mathbf{u}^0$ *of the major player, there is a unique MFG equilibrium* $(\alpha^{\mathbf{u}^0}, m^{\mathbf{u}^0})$ *associated to* $\mathbf{u}^0$ *given by*

$$\alpha_t^{\mathbf{u}^0} = -H_\xi(X_t^*, D\phi_t(X_t^*), \mathbf{p}_t^{\mathbf{u}^0}) \tag{13}$$

*with* $X_t^* = Z - \int_0^t H_\xi(X_s^*, D\phi_s(X_s^*), \mathbf{p}_s^{\mathbf{u}^0})ds + \sqrt{2}B_t$ *and* $m^{\mathbf{u}^0} = m$*, where* $(\phi, m, M)$ *is the unique solution to the MFG system 14 associated to* $(\mathbf{p}^{\mathbf{u}^0})$ *on* $(\Omega^0 \times \mathcal{U}^0, \mathcal{B}(\Omega^0 \times \mathcal{U}^0), (\mathcal{F}_t^{\mathbf{u}^0}), P^{\mathbf{u}^0})$

As mentioned in the text, this greatly simplifies the major players problem. The sup can be ignored as any action they take will only result in a single equilibrium of the small players.

# 10 The Relaxed Problem

Although the major player's problem is greatly simplified, to ensure the inf exists, a relaxation on the action space must occure. Consider:

**Definition 10.1** (The Relaxed Problem)**.** *Recalling the space of measures 5.5, consider the problem*

$$\min_{\mathbf{P}\in\mathbf{M}(p_0)} \bar{J}^0(\mathbf{P}) \quad \text{where} \quad \bar{J}^0(\mathbf{P}) := \mathbb{E}^{\mathbf{P}}\left[\int_0^T \min_{u^0\in U^0} L^0(s, u^0, m_s^{\mathbf{P}}, p_s)ds\right]$$

*where* $(\phi^{\mathbf{P}}, m^{\mathbf{P}}, M^{\mathbf{P}})$ *is the unique solution to 14 on* $(\mathbf{D}, \mathcal{G}, \mathbf{P}, (\mathcal{F}_t^{\mathbf{P}}))$*. That is, the system:*

$$\begin{cases} d\phi_t^{\mathbf{P}}(x) = \{-\Delta\phi_t^{\mathbf{P}}(x) + H(x, D\phi_t^{\mathbf{P}}(x), p_t) - F(x, m_t^{\mathbf{P}}, p_t)\}dt + dM_t^{\mathbf{P}}(x) & \text{in } (0,T)\times\mathbb{R}^d \\ dm_t^{\mathbf{P}}(x) = \{\Delta m_t^{\mathbf{P}}(x) + div(H_\xi(x, D\phi_t^{\mathbf{P}}(x), p_t)m_t^{\mathbf{P}}(x))\}dt & \text{in } (0,T)\times\mathbb{R}^d \\ m_0^{\mathbf{P}}(x) = \bar{m}_0 s(x), \quad \phi_T^{\mathbf{P}}(x) = G(x, m_T^{\mathbf{P}}, p_T) & \text{in } \mathbb{R}^d \end{cases} \tag{14}$$

Although the above problem looks quite different than the original MFG one, the next few proposition reveals that it can guarantee getting $\epsilon$ close to an optimal action for the major player.

**Proposition 10.1** (3.4 in [4])**.** *Let* $\mathbf{u}^0 \in (\Delta(\mathcal{U}^0))^I$*,* $\mathbf{p}^{\mathbf{u}^0}$ *be given as the best option as above and* $\mathbf{P}$ *its law on* $\mathbf{D}$*. Then*

$$\bar{J}^0(\mathbf{P}) \le J^0(\mathbf{u}^0)$$

*where* $J^0$ *is defined above. Further, given a* $\mathbf{P} \in \mathbf{M}(p_0)$ *there exists a sequence* $\bar{u}^{0,n} \in (\Delta(\mathcal{U}^0))^I$ *such that*

$$\lim_n J^0(\mathbf{u}^{0,n}) = \bar{J}^0(\mathbf{P})$$

**Proposition 10.2** (3.5 in [4])**.** *Under all the above assumptions, the minimizer in* $\bar{J}^0$ *exists*

*Proof.* See □

**Corollary 10.1** (Existence of $\epsilon-$minimizer in 9.2)**.** $\forall\epsilon > 0\ \exists\mathbf{u}^{0,\epsilon} \in (\Delta(\mathcal{U}^0))^I$ *such that*

$$J^0(\mathbf{u}^{0,\epsilon}) \le \inf_{\mathbf{u}^0\in(\Delta(\mathcal{U}^0))^I} J^0(\mathbf{u}^0) + \epsilon$$

*Proof.* Let $\mathbf{P}^*$ be the minimizer given by 10.2. By the first half of 10.1 we know necessarily that $\forall\mathbf{u}^0 \in (\Delta(\mathcal{U}^0))^I$ with $\mathbf{p}^{\mathbf{u}^0}$ and $\mathbf{P}$ its law,

$$\bar{J}^0(\mathbf{P}^*) \le \bar{J}^0(\mathbf{P}) \le J^0(\mathbf{u}^0)$$

And so

$$\bar{J}^0(\mathbf{P}^*) \leq \inf_{\mathbf{u}^0 \in (\Delta(\mathcal{U}^0))^I} J^0(\mathbf{u}^0)$$

As $\mathbf{P}^* \in \mathbf{M}(p_0)$ by assumption, the second half of 10.2 gives a $\mathbf{u}^{0,\epsilon}$ such that

$$\begin{aligned} J^0(\mathbf{u}^{0,\epsilon}) &\leq \bar{J}^0(\mathbf{P}^*) + \epsilon \\ &\leq \inf_{\mathbf{u}^0 \in (\Delta(\mathcal{U}^0))^I} J^0(\mathbf{u}^0) + \epsilon \end{aligned}$$

□

Although the optimal action from the major player has not been found, some clever compactness arguments from [22] ensure that the above approximation is enough to get close to optimal in the $N-$player game. The following section outlines this.

# 11 Applications To Finitely Many Players

Consider a space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ supporting $N$ small players with controls $(\alpha^{N,j})_{j=1,\dots,N}$ in the Stackelberg setting 1 from the introduction. They are assumed to have symmetric costs given by

$$J^{N,j}(\mathbf{u}^0, \alpha^{N,j}, (\alpha^{N,k})_{k\neq j}) = \mathbb{E}\left[\int_0^T L_{\mathbf{i}}(X_t^{N,j}, \alpha_t^{N,j}) + F_{\mathbf{i}}(X_t^{N,j}, m^{N,j}_{\mathbf{X}_t^N})dt + G_{\mathbf{i}}(X_T^{N,j}, m^{N,j}_{\mathbf{X}_t^N})\right]$$

where $m^{N,j}_{\mathbf{X}_t^N} = \frac{1}{N-1}\sum_{k\neq j} \delta_{X_t^{N,k}}$
An approximate solution to this problem is a

**Definition 11.1** ($\delta-$Nash Equilibrium). *Fixing $\mathbf{u}^0$, for $\delta > 0$ a $\delta-$Nash equilibrium for the small players are controls such that for $\alpha$ and $j = 1, ..., N$*

$$J^{N,j}(\mathbf{u}^0, \alpha, (\alpha^{N,k}_{k\neq j})) \geq J^{N,j}(\mathbf{u}^0, \alpha^{N,j}, (\alpha^{N,k})_{k\neq j}) - \delta$$

In this $N-$small player game, assume 7 again for the cost functions of the players. Further, assume that the initial distribution $\bar{m}_0 \in \mathcal{P}_1(\mathbb{R}^d)$ has a smooth and bounded density and a finite fourth moment. With this, the authors get the following three results:

**Lemma 11.1.** *Under these assumptions, $\forall\delta > 0$ $\exists N_\delta \in \mathbb{N}$ such that, $\forall \mathbf{u}^0 \in (\Delta(\mathcal{U}^0))^I$ and $N \geq N_\delta$ there exists a $\delta-$Nash equilibrium for $(J^{N,j}(\mathbf{u}^0, \cdot))$*

**Proposition 11.1.** *Proposition 2 (4.3 in [4]) Under the above assumptions with $\epsilon > 0$, there exists $\delta > 0$ and $N'_\delta \geq N_\delta$ such that for any control $\mathbf{u}^0$, $N \geq N'_\delta$, and $\delta-$Nash equilibrium $(\alpha^{N,j})$ for $\mathbf{u}^0$ satisfies*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\mathbf{d}_1(m^N_{\mathbf{X}_t^N}, m_t^{\mathbf{u}^0})\right] \leq \epsilon$$

*where $m^N_{\mathbf{X}_t^N} = \frac{1}{N}\sum_{k=1}^N \delta_{X_t^{N,k}}$ and $(\phi^{\mathbf{u}^0}, \mathbf{m}^{\mathbf{u}^0}, \mathbf{M}^{\mathbf{u}^0})$ is the solution to the MF system associated to $(\mathbf{p}^{\mathbf{u}^0})$ (the information process we defined earlier).*

The combination of the above yields the final result of the paper:

**Theorem 11.1** ($N-$Player Optimality). *Fix $\epsilon > 0$ and let $\bar{\mathbf{u}}^0$ be an $\epsilon-$minimizer for 9.2 provided by 10.1. Then $\exists\delta > 0$ and $N_0 \in \mathbb{N}$ such that, $\forall N \geq N_0$, $\mathbf{u}^0$ is $3\epsilon-$optimal for $J^{N,0}(\cdot, \delta)$*

That is, the optimal control for the major player in the MFG setting is approximately the optimal control in the $N-$player setting assuming the small players are acting close to optimally.

# 12 Extensions and Conclusion

There are three extensions to the work above which would add to the applicability of the setting and the current literature of partial information mean field games. The first one being an extension that the authors mention. That is to expand the allowed actions of the small players. The current setup only allows for open-loop controls adapted to the realization of the major player's control. Allowing for more complicated dependence, for instance the closed-loop setting, would allow the smaller players to react to themselves. This would expand the applicability of the work. The authors mention that Lacker and Le Flem have already done the heavy lifting in the ability to bring MFG equilibria to the $N-$player setting [23]. However, the technicalities present in the relaxed problem would require ironing out. The second possible extension would be to allow for a more complex state space for nature. For instance taking $I = \mathbb{R}$ or another continuum. This of course would expand the applicability of the model to many settings including finance. However, many of the proofs would have to be entirely reworked. Finally, it is important to note that in [4] the major agent does not have their own process. This would result in difficulty when modeling situations in which the major player and small players are supposedly interacting within the same environment. Unfortunately, the canonical hedge fund example is a representation of this. The hedge fund and small traders are trading in the same market. Thus, the decision of a fund to transfer wealth around a market would have large effects on the wealth of a small player. The present setup precludes the major player to more of an observer in the market. In a similar setup, the major agent could have a process that affects the costs and dynamics of the small agents. Depending on the exact details of the major players process, it could be viewed as a noisey observation of the initial state chosen by nature allowing for applications of nonlinear filtering in the decision making of the small players such as in [17].

sIn conclusion, Bergault, Cardaliaguet, and Rainer made an interesting contribution to the developing theory of partial information mean field games. The use of the Stackelberg equilibrium allowed for a natural interpretation of the game solved. Further, the paper connected some of the authors' quite diverse previous work ([6] and [7]). To expand on the above, various extensions can be made to this work that would increase it's applicability. Most pertinently, however, the paper points to the large amount that can still be done to relax assumptions present in the MFG framework. This includes perfect information, but also assumptions like rational expectations among players and the concept of equilibrium itself. Further, there are still many ideas from Dynamic Games that could still yet be realized in the MFG setting.

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# A Mathematical Background

**Lemma A.1** (Lemma 3.4,3.5 from [5]). *Let $m$ be defined as above. There are constants $c_0 = c_0(T)$ and $c_1 = c_1(T)$ such that*

$$\boldsymbol{d}_1(m(t), m(s)) \le c_0(1 + \|b\|_\infty)|t-s|^{\frac{1}{2}} \quad \forall s,t \in [0,T]$$

*and*

$$\int_{\mathbb{R}^d} |x|^2 dm(t)(x) \le c_1 \left( \int_{\mathbb{R}^d} |x|^2 dm_0(x) + 1 + \|b\|_\infty^2 \right) \quad \forall t \in [0,T]$$

*Proof.* Let $\{X_t\}$ be a process satisfying the SDE above. The process at $s,t \in [0,T]$ will have measures $m(s)$ and $m(t)$ respectively by 4.4. Thus, the joint law of $(X_t, X_s) \in \Pi(m(t), m(s))$ and so if $s < t$

$$\begin{aligned}
\mathbf{d}_1(m(t), m(s)) &\le \int_{\mathbb{R}^{2d}} |x-y| d\gamma(x,y) \\
&= \mathbb{E}|X_t - X_s| \\
&\le \mathbb{E}\left[ \int_s^t |b(X_\tau, \tau)| d\tau + \sqrt{2}|B_t - B_s| \right] \\
&\le \|b\|_\infty (t-s) + \sqrt{2(t-s)}
\end{aligned}$$

Further,

$$\begin{aligned}
\int_{\mathbb{R}^d} |x|^2 dm(t)(x) &= \mathbb{E}[|X_t|^2] \\
&\le 2\mathbb{E}\left[ |X_0|^2 + \left| \int_0^t b(X_\tau, \tau) d\tau \right|^2 + 2|B_t|^2 \right] \\
&\le 2\left[ \int_{\mathbb{R}^d} |x|^2 dm_0(x) + t^2 \|b\|_\infty^2 + 2t \right] \\
&\le 2c_1 \left[ \int_{\mathbb{R}^d} |x|^2 dm_0(x) + 1 + \|b\|_\infty^2 \right]
\end{aligned}$$

where $c_1$ is some constant dependent on $T$. □

**Theorem A.1** (Schauder's Fixed Point Theorem [16]). *Suppose $K \subset X$ is nonempty, compact and convex, and assume*

$$A : K \to K$$

*is continuous. Then $A$ has a fixed point in $K$.*

# B Background From Previous Papers

## B.1 Mean Field Games with Common Noise and Degenerate Idiosyncratic Noise

In [7], Cardalaiaguet, Seeger, and Souganidis investigate modified notions of weak solutions of the following MFG system:

$$\begin{cases} du_t = \big[-\beta\,\Delta u_t - \mathrm{tr}\big(a_t(x,m_t)D^2u_t\big) + H_t(x,Du_t,m_t) - 2\beta\,\mathrm{div}(v_t)\big]\,dt \\ \qquad\quad + \, v_t \cdot \sqrt{2\beta}\,dW_t, \quad \text{in } [0,T) \times \mathbb{R}^d, \\ dm_t(x) = \mathrm{div}\Big\{\beta\,Dm_t(x) + \mathrm{div}\big(a_t(x,m_t)\,m_t\big) + m_t\,D_pH_t(x,Du_t,m_t)\Big\}\,dt \\ \qquad\quad - \,\mathrm{div}\big(m_t\sqrt{2\beta}\,dW_t\big), \quad \text{in } (0,T] \times \mathbb{R}^d, \\ m_0(x) = \bar{m}_0(x), \quad u_T(x) = G\big(x,m_T\big), \quad \text{in } \mathbb{R}^d. \end{cases} \tag{15}$$

which is very close to a shifted version of 5. Indeed the change of variables given by

$$\tilde{u}_t(x) = u_t(x + \sqrt{2\beta}W_t) \text{ and } \tilde{m}_t = (Id - \sqrt{2\beta}W_t)_{\#}m_t$$

and similarly shifting all other functions of $x$ in this way (denoting their transforms by a tilde as well) yields an MFG that is beginning to reflect 14. The new system becomes

$$\begin{cases} d\tilde{u}_t = \{-tr(\tilde{a}_t(x,\tilde{m}_t)D^2\tilde{u}_t) + \tilde{H}_t(x,D\tilde{u}_t,\tilde{m}_t)\}dt + dM_t \quad \text{in } [0,T) \times \mathbb{R}^d \\ d\tilde{m}_t(x) = div\{div(\tilde{a}_t(x,\tilde{m}_t)\tilde{m}_t) + \tilde{m}_tD_p\tilde{H}_t(x,D\tilde{u}_t,\tilde{m}_t)\}dt \quad \text{in } (0,T] \times \mathbb{R}^d \\ \tilde{m}_0(x) = \bar{m}_0(x), \quad \tilde{u}_T(x) = \tilde{G}(x,\tilde{m}_T) \quad \text{in } \mathbb{R}^d \end{cases} \tag{16}$$

The solution to the backward stochastic HJB equation is said to be weak in the PDE sense and strong in the probabilistic one [7]. In particular, Definition 3.1 of [7] states the specific definition of this type of solution.

**Theorem B.1** (4.1 in [7])**.** *Assuming essentially the same assumptions as seen above, there exists a weak solution of 16 with respect to the Weiner measure.*

Weak solutions can be turned into strong when pathwise uniqueness holds. With several assumptions, this in fact holds with our system.

**Assumptions B.1.** *Consider the assumptions:*

- *The Hamiltonian is separable, that is,*

$$H_t(x,\xi,m) = H_t(x,\xi) - F_t(x,m)$$

- *And $F$ and $G$ are strictly monotone i.e. $\forall m, m' \in \mathcal{P}_2(\mathbb{R}^d)$ and $t \in [0,T]$*

$$\begin{cases} \displaystyle\int_{\mathbb{R}^d} (F_t(x,m) - F_t(x,m'))(m-m')(dx) \geq 0 \\ \displaystyle\int_{\mathbb{R}^d} (G(x,m) - G(x,m'))(m-m')(dx) \geq 0 \end{cases} \tag{17}$$

  *with equality holding if and only if $m = m'$*

- *$a_t(x,m) = a_t(x)$ (independence from $m$)*

Under another transformation, 16 then becomes

$$\begin{cases} d\tilde{u}_t = \{-tr(\tilde{a}_t(x)D^2\tilde{u}_t) + \tilde{H}_t(x,D\tilde{u}_t) - \tilde{F}_t(x,\tilde{m}_t)\}dt + dM_t \quad \text{in } (0,T) \times \mathbb{R}^d \\ d\tilde{m}_t(x) = div\{div(\tilde{a}_t(x)\tilde{m}_t) + \tilde{m}_tD_p\tilde{H}_t(x,D\tilde{u}_t)\}dt \quad \text{in } (0,T) \times \mathbb{R}^d \\ \tilde{m}_0(x) = \bar{m}_0(x), \quad \tilde{u}_T(x) = \tilde{G}(x,\tilde{m}_T) \quad \text{in } \mathbb{R}^d \end{cases} \tag{18}$$

**Theorem B.2.** *Assuming B.1 we get that pathwise uniqueness holds and therefore by Yamada-Watanabe 4.2, every weak solution is a strong solution.*